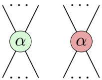
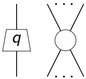
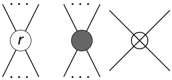
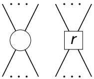
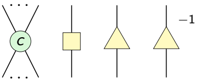
Phase Ring Calculi

Hector Miller-Bakewell

University of Oxford

2019-12-09

Context

ZX		$\mathbb{C}$-bit	Groups??
ZQ		$\mathbb{C}$-bit	Groups?
ZW_R		R-bit	Rings
$\mathrm{ZH}_{R[\frac{1}{2}]}$		R-bit	Rings
AZX		$\mathbb{C}$-bit	Rings

The plan

- Introduce a new calculus built from ring operations

The plan

- Introduce a new calculus built from ring operations
- Say something new about ZW (Errata: "Well known, but not written down anywhere I could find" is more accurate)

The plan

- Introduce a new calculus built from ring operations
- Say something new about ZW (Errata: "Well known, but not written down anywhere I could find" is more accurate)
- Look at what phase-ring calculi are

The plan

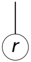
- Introduce a new calculus built from ring operations
- Say something new about ZW (Errata: "Well known, but not written down anywhere I could find" is more accurate)
- Look at what phase-ring calculi are
- Relate this new calculus to qubit phase-ring calculi

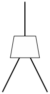
The (compact closed) graphical calculus $\textsc{Ring}_R$

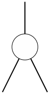
For R a commutative ring, interpreting into R-bits:

The (compact closed) graphical calculus $\textsc{Ring}_R$

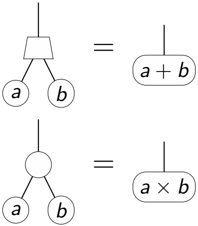
For R a commutative ring, interpreting into R-bits:

states  $\begin{pmatrix} 1 \\ r \end{pmatrix}$

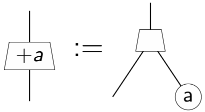
addition  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

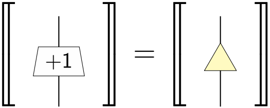
multiplication  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

It does act like a ring

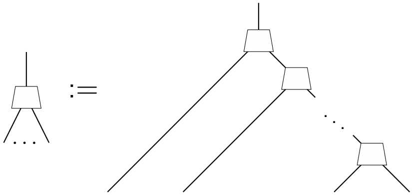


Derived generators

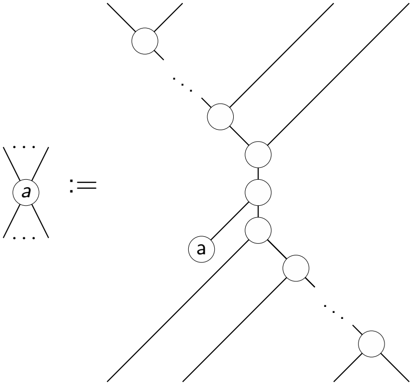




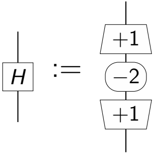
$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



Derived generators

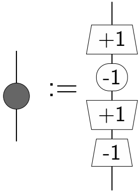


Derived generators



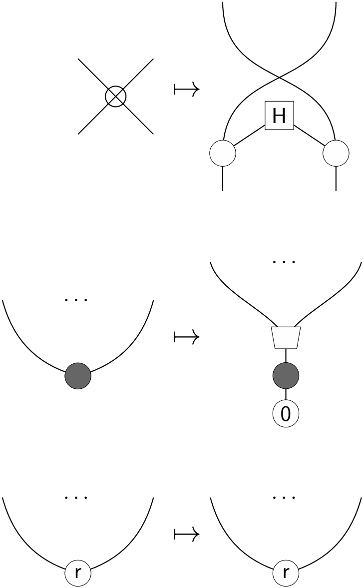
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Derived generators

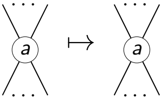


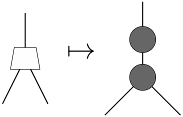
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

From ZW

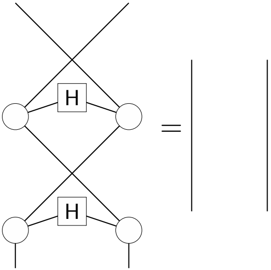


To ZW

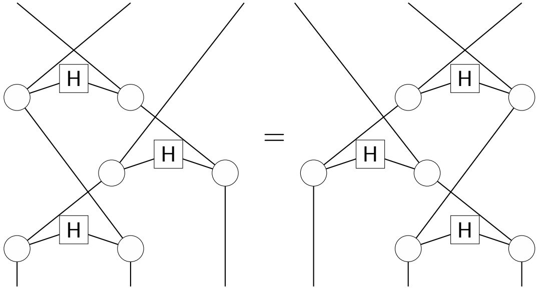




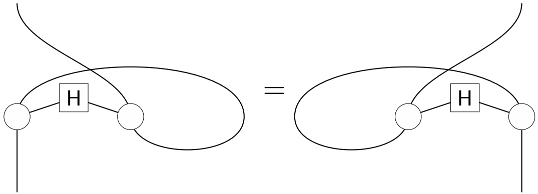
Translation of rei_2^x from ZW



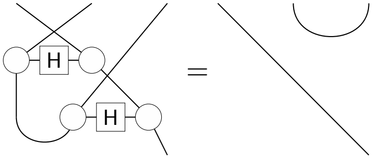
Translation of rei_3^x from ZW



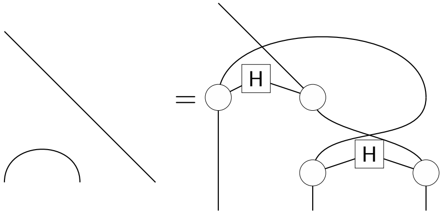
Translation of rei_1^x from ZW



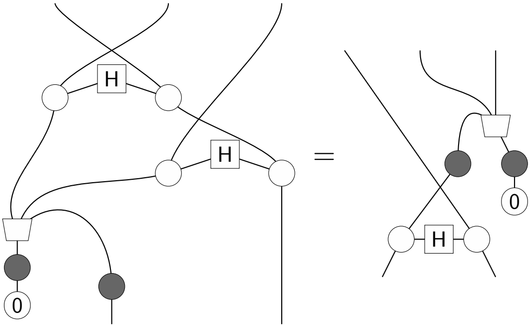
Translation of nat_x^η from ZW



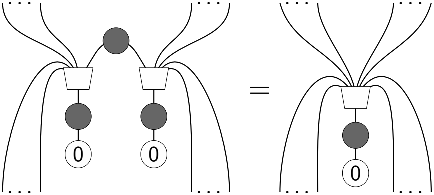
Translation of nat_x^ϵ from ZW



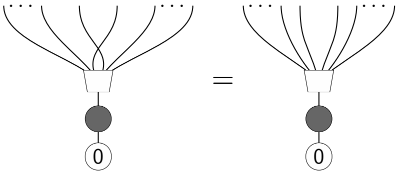
Translation of nat_x^w from ZW



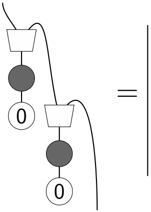
Translation of cut_w from ZW



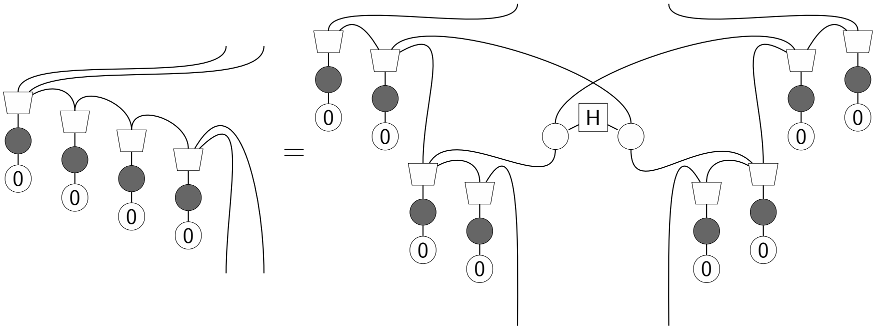
Translation of tr_w from ZW



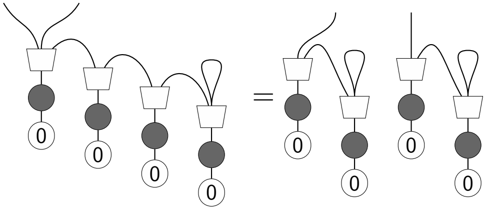
Translation of *inv* from ZW



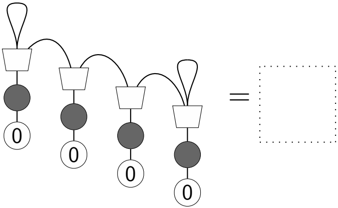
Translation of 5a from ZW



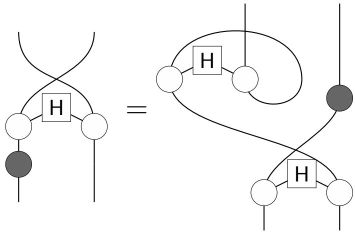
Translation of 5b from ZW



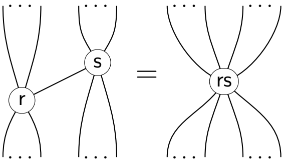
Translation of 5c from ZW



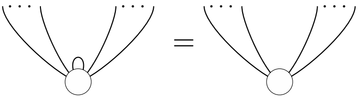
Translation of ant_x^n from ZW



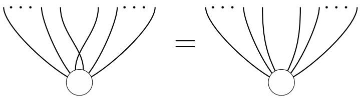
Translation of cut_z from ZW



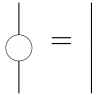
Translation of tr_z from ZW



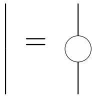
Translation of sym_z from ZW



Translation of *id* from ZW

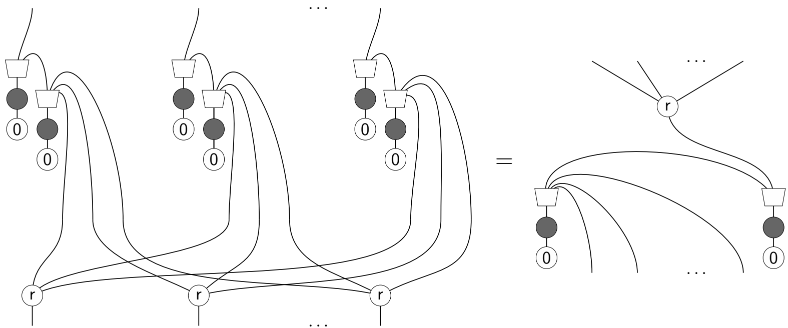


Translation of rng_1 from ZW

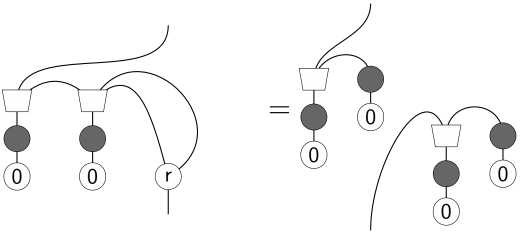


Translation of ba_{zw} from ZW

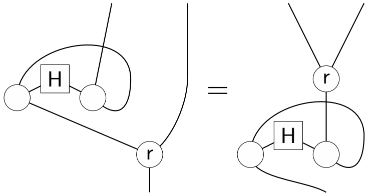
Note that there must be at least 1 output



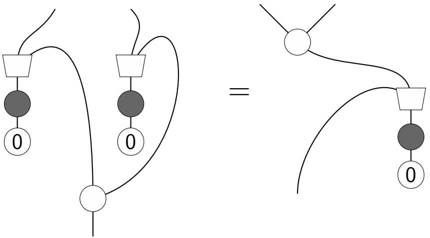
Translation of *loop* from ZW



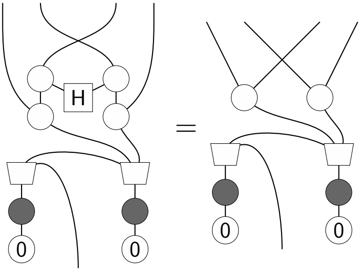
Translation of *ph* from ZW



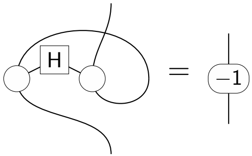
Translation of nat_c^n from ZW



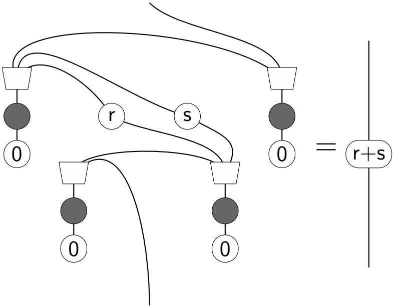
Translation of *unx* from ZW



Translation of rng_{-1} from ZW

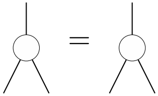


Translation of $rng_{+}^{r,s}$ from ZW

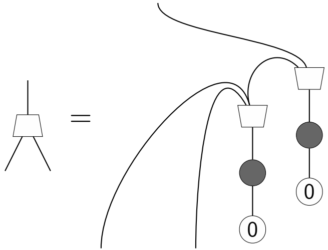


Finally, the re-translated generators

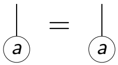
Translation of multiplication



Translation of addition



Translation of states



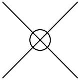
Done!

So RING_R is universal and complete for R-bits

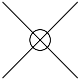
So $\textsc{ring}_R$ is universal and complete for R-bits

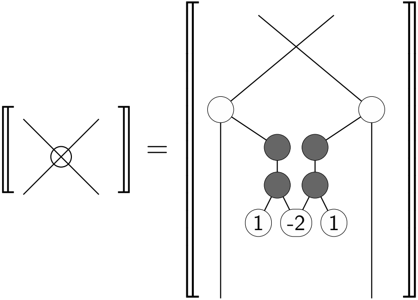
And just uses ring operations and compact closure

Crossing

Note this also gives us a spider version of  !

Crossing

Note this also gives us a spider version of  !



Crossing

So the crossing is now redundant in ZW

Summary of first part

- RING_R is universal and complete for R-bits, made just from $+$, $\times$, states and compact closure

Summary of first part

- $\textsc{Ring}_R$ is universal and complete for R-bits, made just from $+$, $\times$, states and compact closure
- ZW_R is now a spider language

How did I get here?

What are phase ring graphical calculi?

How did I get here?

What are phase ring graphical calculi?

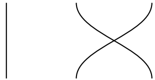
Start with a PROP

The PROP RING R

We define the PROP $\textsc{ring-prop}_R$ for the (not necessarily commutative) ring R as generated by:

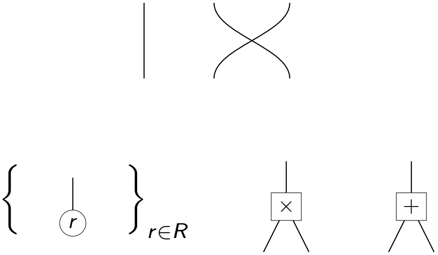
The PROP RING R

We define the PROP $\textsc{ring-prop}_R$ for the (not necessarily commutative) ring R as generated by:



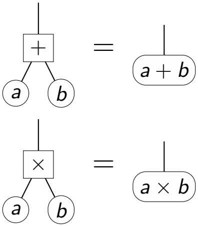
The PROP RING R

We define the PROP $\textsc{ring-prop}_R$ for the (not necessarily commutative) ring R as generated by:



Some Ring-like rules

We will require these rules to be sound:



An interpretation

We will require $\textsc{ring-prop}_R$ to have an interpretation into K-bits, where K is a field

An interpretation

We will require $\textsc{ring-prop}_R$ to have an interpretation into K-bits, where K is a field

$$\begin{aligned} [\![\cdot]\!] : \textsc{ring-prop}_R &\to \mathsf{Mat}_K \\ D_m^n &\mapsto M \in \mathsf{Mat}_{2^m \times 2^n}(K) \end{aligned}$$

An interpretation

We will require $\textsc{ring-prop}_R$ to have an interpretation into K-bits, where K is a field

$$\begin{aligned} [\![\cdot]\!] : \textsc{ring-prop}_R &\to \mathsf{Mat}_K \\ D_m^n &\mapsto M \in \mathsf{Mat}_{2^m \times 2^n}(K) \end{aligned}$$

This interpretation needs to be faithful on the generators of $\textsc{ring-prop}_R$.

An interpretation

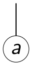
We will require $\textsc{ring-prop}_R$ to have an interpretation into K-bits, where K is a field

$$\begin{aligned}[\![\cdot]\!] : \textsc{ring-prop}_R &\to \mathsf{Mat}_K \\ D^n_m &\mapsto M \in \mathsf{Mat}_{2^m \times 2^n}(K)\end{aligned}$$

This interpretation needs to be faithful on the generators of $\textsc{ring-prop}_R$.

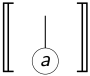
Call $\textsc{ring-prop}_R$ with this interpretation $\textsc{ring-prop}_{R/K}$.

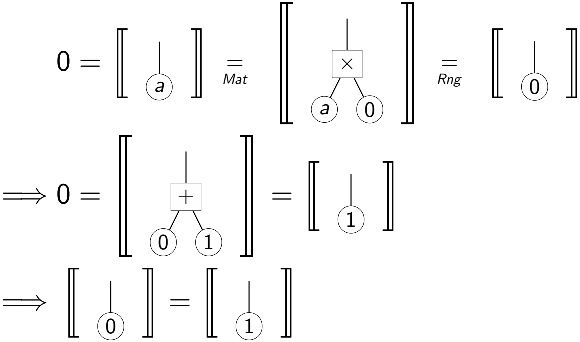
Non-zero

Faithfulness implies $\llbracket$ 

$\rrbracket \neq 0$:

Non-zero

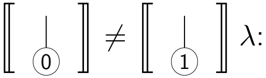
Faithfulness implies  $\neq 0$:

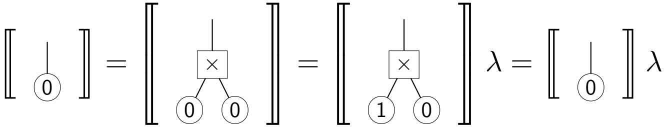


Non-colinear

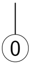
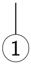
Since K is an integral domain then $\left[\!\!\left[\;\overset{|}{\textcircled{0}}\; \right]\!\!\right] \neq \left[\!\!\left[\;\overset{|}{\textcircled{1}}\; \right]\!\!\right] \lambda$:

Non-colinear

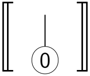
Since K is an integral domain then 

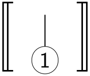


Basis

So $\llbracket$  $\rrbracket$ and $\llbracket$  $\rrbracket$ are non-zero, non-colinear vectors in K^2.

Basis

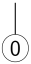
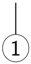
So 

and 

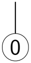
are non-zero, non-colinear vectors in K^2.

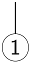
So form a basis!

Basis

So $[\![$  $]\!]$ and $[\![$  $]\!]$ are non-zero, non-colinear vectors in K^2.

So form a basis!

$[\![$  $]\!] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$[\![$  $]\!] = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Not quite physics

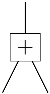
It's not the $|0\rangle$ and $|1\rangle$ basis, but bear with me.

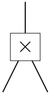
Interpretation of plus and times

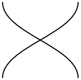
This choice of basis for the interpretation of $\textsc{ring-prop}_{R/K}$ forces the following interpretations:

Interpretation of plus and times

This choice of basis for the interpretation of RING-PROP$_{R/K}$ forces the following interpretations:

$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$[\![$  $]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Proof of interpretation

These equations are true for rings, so need to be true graphically as well:

$$0 \times 0 = 0 \qquad 0 \times 1 = 1$$
$$1 \times 0 = 0 \qquad 1 \times 1 = 1$$
$$0 + 0 = 0 \qquad 0 + 1 = 1$$
$$1 + 0 = 1 \qquad ???$$

Proof of interpretation

These equations are true for rings, so need to be true graphically as well:

$$0 \times 0 = 0 \qquad 0 \times 1 = 1$$
$$1 \times 0 = 0 \qquad 1 \times 1 = 1$$
$$0 + 0 = 0 \qquad 0 + 1 = 1$$
$$1 + 0 = 1 \qquad ???$$

$$(1 + 1) \times 0 = 0 \qquad (1 + 1) \times (1 + 1) = (1 + 1) + (1 + 1)$$

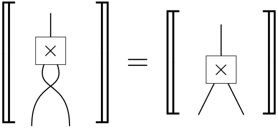
(Recalling that we don't know char K.)

Commutativity

Note that this interpretation forces commutativity of R

Commutativity

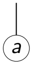
Note that this interpretation forces commutativity of R



Limitations

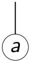
What other limitations would $\textsc{ring-prop}_{R/K}$ impose on R?

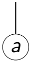
Interpreting states

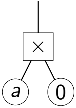
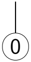
Let's look at a generic state 

in RING-PROP$_{R/K}$:

Interpreting states

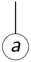
Let's look at a generic state  in RING-PROP$_{R/K}$:

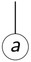
$[\![$  $]\!] = \begin{pmatrix} b \\ c \end{pmatrix}$

$\implies [\![$  $]\!] = \begin{pmatrix} b \\ 0 \end{pmatrix} = [\![$  $]\!] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\implies b = 1$

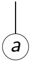
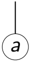
First element is 1

So for every state 

of $\textsc{ring-prop}_{R/K}$, $\left[\!\left[\ \right.\right.$ 

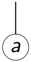
$\left.\left.\ \right]\!\right]_1 = 1$

First element is 1

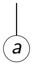
So for every state  of RING-PROP$_{R/K}$, $\llbracket$  $\rrbracket_1 = 1$

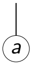
So there is a function of sets

$$f : R \to K$$

$a \mapsto \llbracket$  $\rrbracket_2$

First element is 1

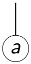
So for every state 

of $\textsc{ring-prop}_{R/K}$, $[\![$ 

$]\!]_1 = 1$

So there is a function of sets

$$f : R \to K$$

$a \mapsto [\![$ 

$]\!]_2$

This must be injective, because $[\![\cdot]\!]$ is faithful

Function of rings

But f is also a ring homomorphism (easy to show because of choice of basis)

Function of rings

But f is also a ring homomorphism (easy to show because of choice of basis)

So $f : R \hookrightarrow K$ as rings

Classifying ring substructures

If you can exhibit $\textsc{ring-prop}_{R/K}$ in your graphical language over K-bits

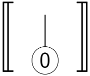
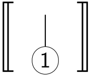
Classifying ring substructures

If you can exhibit $\textsc{ring-prop}_{R/K}$ in your graphical language over K-bits

- That ring is a subring of K

Classifying ring substructures

If you can exhibit $\textsc{ring-prop}_{R/K}$ in your graphical language over K-bits

- That ring is a subring of K
- The interpretation is determined by  and 

Questions

Any questions on $\textsc{ring-prop}_{R/K}$?

Full substructure

What about $\textsc{ring-prop}_{K/K}$?

Universality

Is $\textsc{ring-prop}_{K/K}$ universal?

Universality

Is $\textsc{ring-prop}_{K/K}$ universal?

No. Can't form morphisms of shape $1 \to 0$

Compact closed

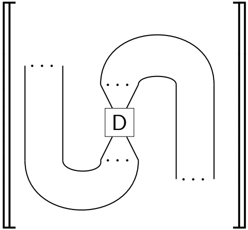
How about if we make $\textsc{ring-prop}_{K/K}$ compact closed?

Compact closed

Define $\textsc{ring-prop}^{T}_{R/K}$ as $\textsc{ring-prop}_{R/K}$ with cups and caps.

Compact closed

Define $\textsc{ring-prop}^T_{R/K}$ as $\textsc{ring-prop}_{R/K}$ with cups and caps.



$:= [\![D]\!]^T$

Back over fields

This requirement on cups and caps in $\text{RING-PROP}^{T}_{R/K}$ forces the usual interpretations.

$$[\![\cup]\!] = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad [\![\cap]\!] = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

Universality

Is $\textsc{ring-prop}^{T}_{K/K}$ universal?

Universality

Is $\textsc{ring-prop}^{T}_{K/K}$ universal?

Yes. Can form all of $\textsc{ring}_K$.

Uniqueness

Every aspect of the interpretation of $\text{RING-PROP}_{R/K}^{T}$ is entirely determined up to choice of basis, and it's universal

Uniqueness

Every aspect of the interpretation of $\textsc{ring-prop}_{R/K}^{T}$ is entirely determined up to choice of basis, and it's universal

It has some sort of uniqueness property.

Phase-Rings as Monoids

The category **PR-GC**(R/K) of Phase-Ring Graphical Calculi (exhibited as monoids) for a ring R interpreted into K-bits has:

Phase-Rings as Monoids

The category **PR-GC**(R/K) of Phase-Ring Graphical Calculi (exhibited as monoids) for a ring R interpreted into K-bits has:

Objects are compact closed PROPs, with additional generators G, an interpretation into K-bits, and distinguished diagrams:

$$\left(G, \llbracket \cdot \rrbracket, \{g_r\}_{r\in R}, g_+, g_\times\right)$$

Phase-Rings as Monoids

The category **PR-GC**(R/K) of Phase-Ring Graphical Calculi (exhibited as monoids) for a ring R interpreted into K-bits has:

Objects are compact closed PROPs, with additional generators G, an interpretation into K-bits, and distinguished diagrams:

$$(G, [\![\cdot]\!], \{g_r\}_{r \in R}, g_+, g_\times)$$

Need addition and multiplication to be sound, and the interpretation to be faithful.

Phase-Rings as Monoids

A morphism

$$f : (G, [\![\cdot]\!], \{g_r\}_{r \in R}, g_+, g_\times) \to (H, [\![\cdot]\!], \{h_r\}_{r \in R}, h_+, h_\times)$$

is a morphism of compact closed props

$$f : G\text{-diagrams} \to H\text{-diagrams}$$

Such that

Phase-Rings as Monoids

A morphism

$$f : (G, [\![\cdot]\!], \{g_r\}_{r\in R}, g_+, g_\times) \to (H, [\![\cdot]\!], \{h_r\}_{r\in R}, h_+, h_\times)$$

is a morphism of compact closed props

$$f : G\text{-diagrams} \to H\text{-diagrams}$$

Such that

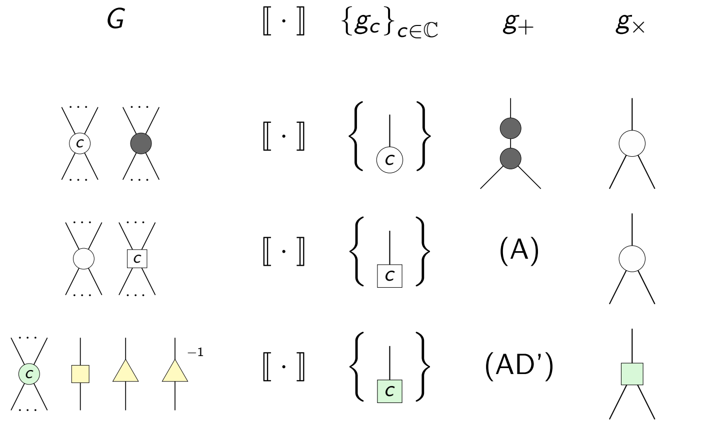
$$f(g_r) = h_r \qquad f(g_+) = h_+ \qquad f(g_\times) = h_\times$$

What's in this category?

Looking at **PR-GC**$(\mathbb{C}/\mathbb{C})$, it includes

What's in this category?

Looking at **PR-GC**($\mathbb{C}/\mathbb{C}$), it includes



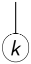
$\textsc{Ring}_K$ initial

$\textsc{Ring}_K$ is initial in **PR-GC**(K/K)

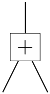
RING_K initial

RING_K is initial in **PR-GC**(K/K)

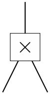
The morphism out of RING_K is easy to construct:

$f($ 

$) := h_k$

$f($ 

$) := h_+$

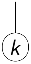
$f($ 

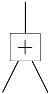
$) := h_\times$

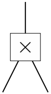
RING_K initial

RING_K is initial in $\mathbf{PR\text{-}GC}(K/K)$

The morphism out of RING_K is easy to construct:

$f($ 

$) := h_k$ $f($ 

$) := h_+$ $f($ 

$) := h_\times$

The morphism is unique because these are all the generators

$\textsc{ring}_{\mathbb{C}}$ initial

In particular, $\textsc{ring}_{\mathbb{C}}$ is initial for qubit graphical calculi which use $\mathbb{C}$ as their phase ring.

$\textsc{ring}_{\mathbb{C}}$ initial

In particular, $\textsc{ring}_{\mathbb{C}}$ is initial for qubit graphical calculi which use $\mathbb{C}$ as their phase ring.

It is universal, sound and complete

$\textsc{ring}_{\mathbb{C}}$ initial

In particular, $\textsc{ring}_{\mathbb{C}}$ is initial for qubit graphical calculi which use $\mathbb{C}$ as their phase ring.

It is universal, sound and complete

Its interpretation is fully determined (up to change of basis)

Summary

- $\textsc{Ring}_R$ is a universal, complete calculus for R-bits, just built from ring operations acting on states

Summary

- $\textsc{ring}_R$ is a universal, complete calculus for R-bits, just built from ring operations acting on states
- The substructures $\textsc{ring-prop}_{R/K}$ have their interpretations determined up to change of basis, and R must be a subring of K

Summary

- $\textsc{ring}_R$ is a universal, complete calculus for R-bits, just built from ring operations acting on states
- The substructures $\textsc{ring-prop}_{R/K}$ have their interpretations determined up to change of basis, and R must be a subring of K
- $\textsc{ring}_{\mathbb{C}}$ is initial in $\mathbf{PR\text{-}GC}(\mathbb{C}/\mathbb{C})$

Thank you!