

Phase Ring Calculi

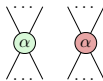
Hector Miller-Bakewell

University of Oxford

2019-12-09

Context

ZX



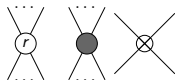
\mathbb{C} -bit Groups??

ZQ



\mathbb{C} -bit Groups?

ZW_R



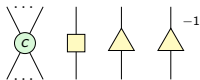
R -bit Rings

$ZH_{R[\frac{1}{2}]}$



R -bit Rings

AZX



\mathbb{C} -bit Rings

The plan

- ▶ Introduce a new calculus built from ring operations

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- ▶ Say something new about ZW (Errata: “Well known, but not written down anywhere I could find” is more accurate)
- ▶ Look at what phase-ring calculi are
- ▶ Relate this new calculus to qubit phase-ring calculi

The (compact closed) graphical calculus RING_R


For R a commutative ring, interpreting into R -bits:

The (compact closed) graphical calculus RING_R

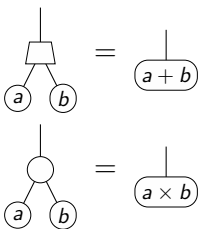
For R a commutative ring, interpreting into R -bits:

states  $\begin{pmatrix} 1 \\ r \end{pmatrix}$

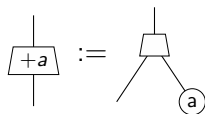
addition  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

multiplication  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

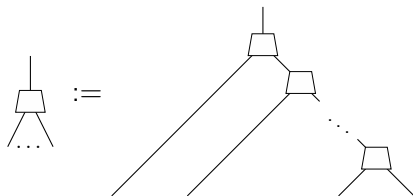
It does act like a ring



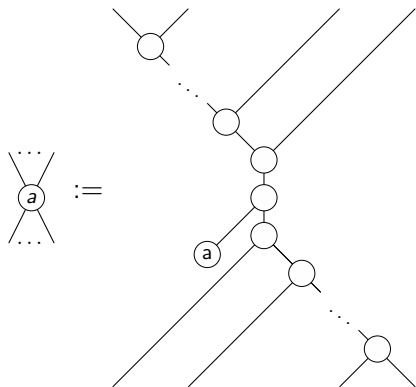
Derived generators



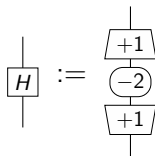
$$\left[\begin{array}{c} \text{trapezoid} \\ +1 \\ \text{trapezoid} \end{array} \right] = \left[\begin{array}{c} \text{trapezoid} \\ \text{yellow triangle} \\ \text{trapezoid} \end{array} \right] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



Derived generators



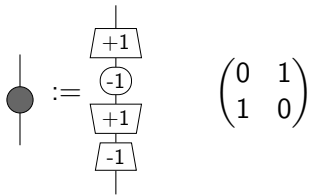
Derived generators



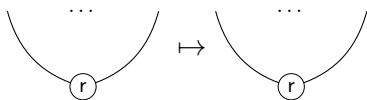
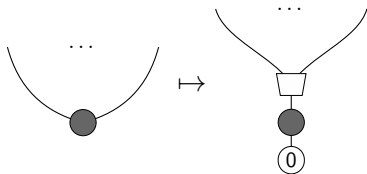
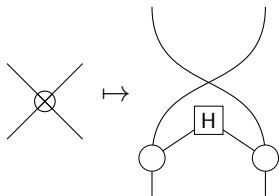
The diagram shows a square box labeled H on the left, connected to a vertical line. This is followed by an equals sign with a double bar ($:=$). To the right of the equals sign is a vertical line with three gates in series: a trapezoid with $+1$ inside, a circle with -2 inside, and another trapezoid with $+1$ inside. To the right of this sequence is a 2x2 matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

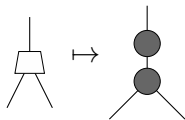
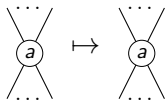
Derived generators



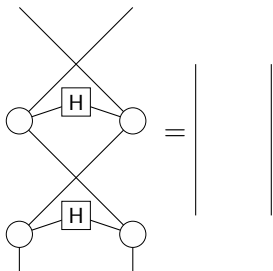
From ZW



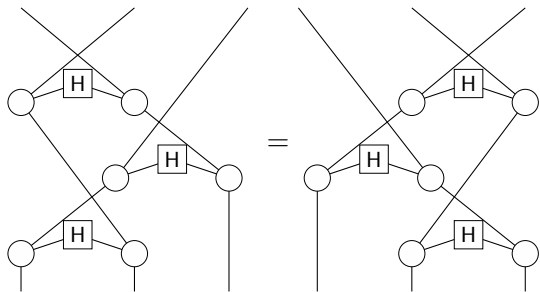
To ZW



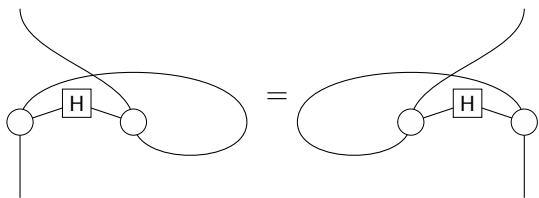
Translation of rei_2^x from ZW



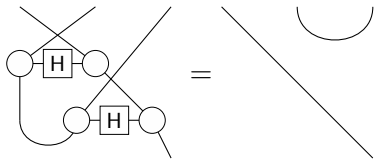
Translation of rei_3^x from ZW



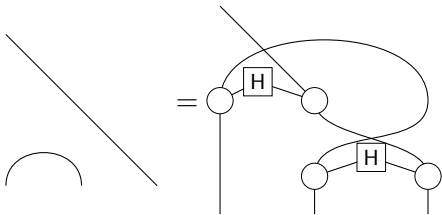
Translation of rei_1^x from ZW



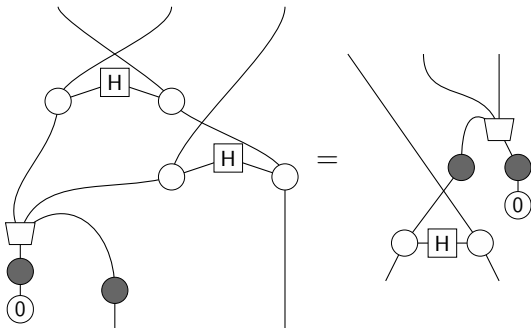
Translation of nat_x^η from ZW



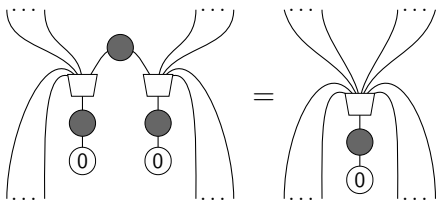
Translation of nat_x^ϵ from ZW



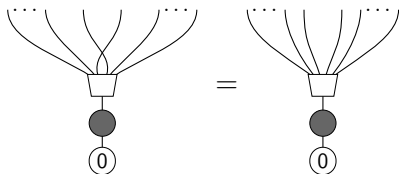
Translation of nat_x^w from ZW



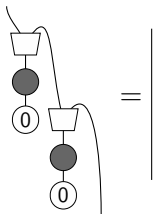
Translation of cut_w from ZW



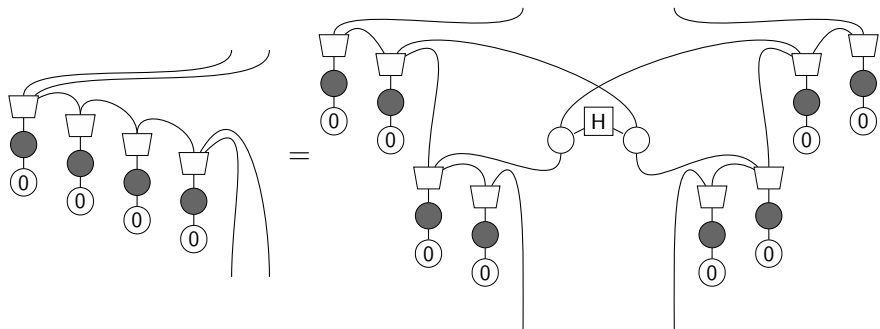
Translation of tr_w from ZW



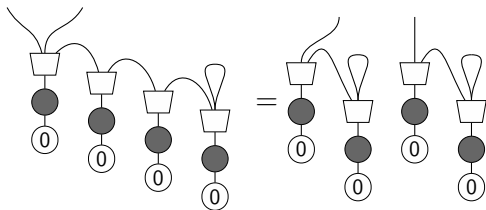
Translation of *inv* from ZW



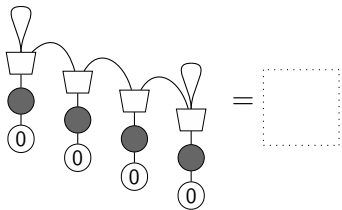
Translation of 5a from ZW



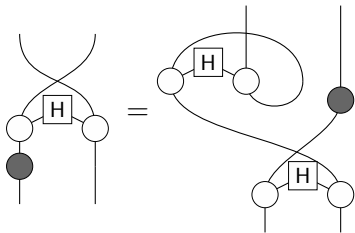
Translation of 5b from ZW



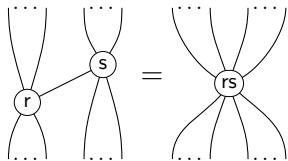
Translation of 5c from ZW



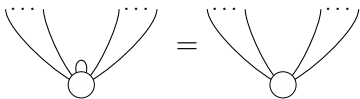
Translation of ant_x^n from ZW



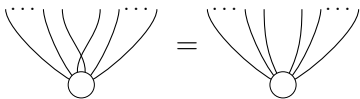
Translation of cut_z from ZW



Translation of tr_z from ZW



Translation of sym_z from ZW



Translation of id from ZW

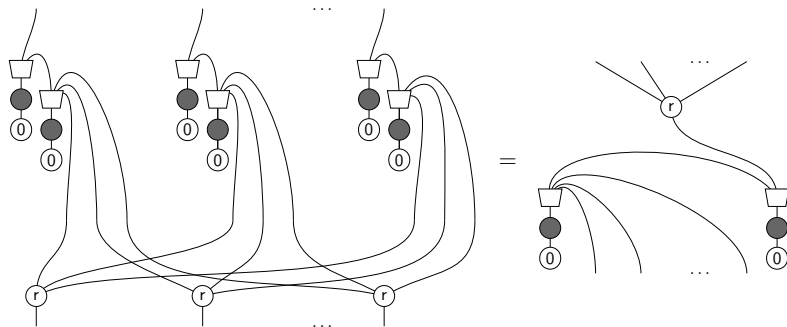


Translation of rng_1 from ZW

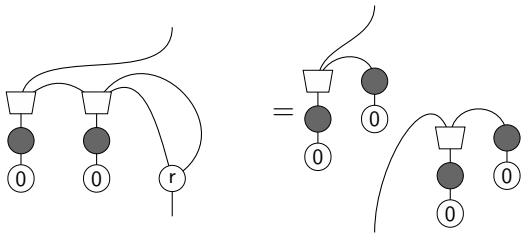


Translation of ba_{ZW} from ZW

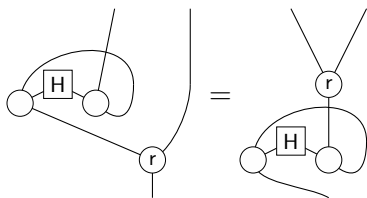
Note that there must be at least 1 output



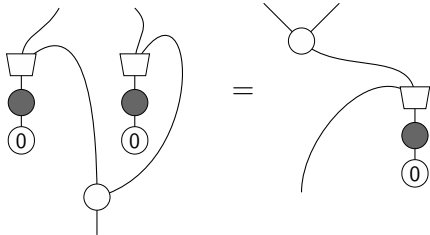
Translation of *loop* from ZW



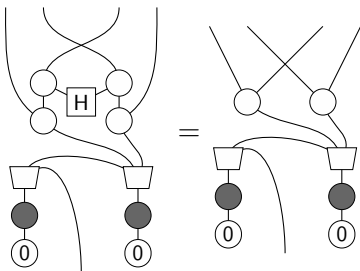
Translation of ph from ZW



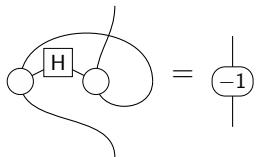
Translation of nat_c^n from ZW



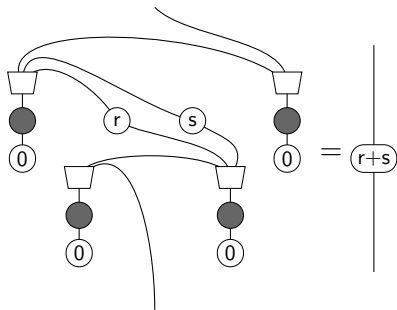
Translation of *unx* from ZW



Translation of rng_{-1} from ZW

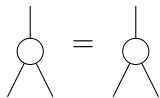


Translation of $rng_+^{r,s}$ from ZW

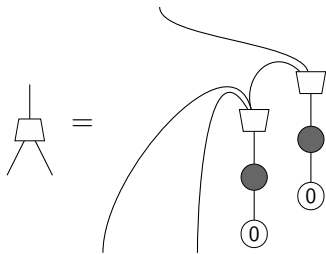


Finally, the re-translated generators

Translation of multiplication



Translation of addition



Translation of states

$$\begin{array}{c} | \\ \circ \\ a \end{array} = \begin{array}{c} | \\ \circ \\ a \end{array}$$


Done!

So RING_R is universal and complete for R -bits


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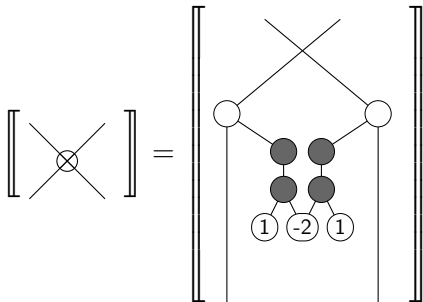
And just uses ring operations and compact closure

Crossing

Note this also gives us a spider version of  !

Crossing

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Crossing

So the crossing is now redundant in ZW

Summary of first part

- ▶ RING_R is universal and complete for R -bits, made just from $+$, \times , states and compact closure

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- ▶ ZW_R is now a spider language

How did I get here?

What are phase ring graphical calculi?

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Start with a PROP

The PROP RING R

We define the PROP RING-PROP_R for the (not necessarily commutative) ring R as generated by:

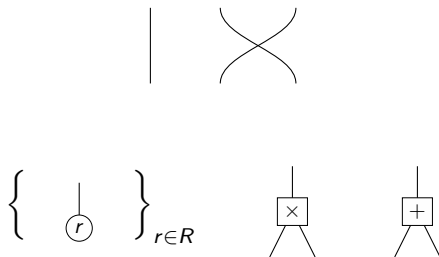
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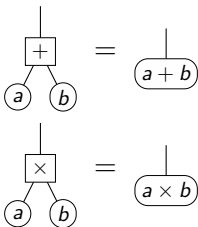
The PROP RING R

We define the PROP RING-PROP_R for the (not necessarily commutative) ring R as generated by:



Some Ring-like rules

We will require these rules to be sound:



An interpretation

We will require RING-PROP_R to have an interpretation into K -bits, where K is a field

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$$\begin{aligned} \llbracket \cdot \rrbracket : \text{RING-PROP}_R &\rightarrow \text{Mat}_K \\ D_m^n &\mapsto M \in \text{Mat}_{2^m \times 2^n}(K) \end{aligned}$$

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This interpretation needs to be faithful on the generators of RING-PROP_R .

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This interpretation needs to be faithful on the generators of RING-PROP_R .

Call RING-PROP_R with this interpretation $\text{RING-PROP}_{R/K}$.

Non-zero

Faithfulness implies $\left[\begin{array}{c} | \\ \circlearrowleft{a} \end{array} \right] \neq 0$:

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$$\begin{aligned} 0 &= \left[\begin{array}{c} | \\ \textcircled{a} \end{array} \right] \stackrel{Mat}{=} \left[\begin{array}{c} | \\ \boxed{\times} \\ \textcircled{a} \quad \textcircled{0} \end{array} \right] \stackrel{Rng}{=} \left[\begin{array}{c} | \\ \textcircled{0} \end{array} \right] \\ \Rightarrow 0 &= \left[\begin{array}{c} | \\ \boxed{+} \\ \textcircled{0} \quad \textcircled{1} \end{array} \right] = \left[\begin{array}{c} | \\ \textcircled{1} \end{array} \right] \\ \Rightarrow \left[\begin{array}{c} | \\ \textcircled{0} \end{array} \right] &= \left[\begin{array}{c} | \\ \textcircled{1} \end{array} \right] \end{aligned}$$

Non-colinear

Since K is an integral domain then $\begin{bmatrix} | \\ \textcircled{0} \end{bmatrix} \neq \begin{bmatrix} | \\ \textcircled{1} \end{bmatrix} \lambda$:

Non-colinear

Since K is an integral domain then $\left[\begin{array}{c} | \\ \textcircled{0} \end{array} \right] \neq \left[\begin{array}{c} | \\ \textcircled{1} \end{array} \right] \lambda$:

$$\left[\begin{array}{c} | \\ \textcircled{0} \end{array} \right] = \left[\begin{array}{c} | \\ \boxed{\times} \\ \textcircled{0} \quad \textcircled{0} \end{array} \right] = \left[\begin{array}{c} | \\ \boxed{\times} \\ \textcircled{1} \quad \textcircled{0} \end{array} \right] \lambda = \left[\begin{array}{c} | \\ \textcircled{0} \end{array} \right] \lambda$$

Basis

So $\begin{bmatrix} | \\ \textcircled{0} \\ | \end{bmatrix}$ and $\begin{bmatrix} | \\ \textcircled{1} \\ | \end{bmatrix}$ are non-zero, non-colinear vectors in K^2 .

Basis

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So form a basis!

Basis

So $\begin{bmatrix} | \\ \textcircled{0} \end{bmatrix}$ and $\begin{bmatrix} | \\ \textcircled{1} \end{bmatrix}$ are non-zero, non-colinear vectors in K^2 .

So form a basis!

$$\begin{bmatrix} | \\ \textcircled{0} \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} | \\ \textcircled{1} \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Not quite physics

It's not the $|0\rangle$ and $|1\rangle$ basis, but bear with me.

Interpretation of plus and times

This choice of basis for the interpretation of $\text{RING-PROP}_{R/K}$ forces the following interpretations:

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$$\left[\begin{array}{c} | \\ | \\ \text{+} \\ | \\ | \end{array} \right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\left[\begin{array}{c} | \\ | \\ \text{x} \\ | \\ | \end{array} \right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left[\begin{array}{c} | \\ | \\ \text{X} \\ | \\ | \end{array} \right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof of interpretation

These equations are true for rings, so need to be true graphically as well:

$$0 \times 0 = 0$$

$$1 \times 0 = 0$$

$$0 + 0 = 0$$

$$1 + 0 = 1$$

$$0 \times 1 = 1$$

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???

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$$1 + 0 = 1$$

???

$$(1 + 1) \times 0 = 0 \quad (1 + 1) \times (1 + 1) = (1 + 1) + (1 + 1)$$

(Recalling that we don't know $\text{char } K$.)

Commutativity

Note that this interpretation forces commutativity of R

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The diagram shows an equality between two expressions. On the left, a square box containing an 'x' is positioned above a cup-shaped line (representing multiplication). This entire structure is enclosed between two vertical bars. On the right, the same square box with 'x' is positioned above a cap-shaped line (representing multiplication). This structure is also enclosed between two vertical bars. An equals sign is placed between the two vertical bar structures.

Limitations

What other limitations would $\text{RING-PROP}_{R/K}$ impose on R ?

Interpreting states

Let's look at a generic state $\begin{array}{c} | \\ \circlearrowleft a \end{array}$ in $\text{RING-PROP}_{R/K}$:

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Let's look at a generic state $\begin{array}{c} | \\ \textcircled{a} \end{array}$ in $\text{RING-PROP}_{R/K}$:

$$\begin{aligned} & \left[\begin{array}{c} | \\ \textcircled{a} \end{array} \right] = \begin{pmatrix} b \\ c \end{pmatrix} \\ \Rightarrow & \left[\begin{array}{c} | \\ \boxed{\times} \\ \textcircled{a} \quad \textcircled{0} \end{array} \right] = \begin{pmatrix} b \\ 0 \end{pmatrix} = \left[\begin{array}{c} | \\ \textcircled{0} \end{array} \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \Rightarrow & b = 1 \end{aligned}$$

First element is 1

So for every state $\begin{array}{c} | \\ \textcircled{a} \end{array}$ of $\text{RING-PROP}_{R/K}$, $\left[\begin{array}{c} | \\ \textcircled{a} \end{array} \right]_1 = 1$

First element is 1

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So there is a function of sets

$$f : R \rightarrow K$$

$$a \mapsto \left[\begin{array}{c} | \\ \circlearrowleft a \end{array} \right]_2$$

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This must be injective, because $[[\cdot]]$ is faithful

Function of rings

But f is also a ring homomorphism (easy to show because of choice of basis)

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So $f : R \hookrightarrow K$ as rings

Classifying ring substructures

If you can exhibit $\text{RING-PROP}_{R/K}$ in your graphical language over K -bits

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If you can exhibit $\text{RING-PROP}_{R/K}$ in your graphical language over K -bits

- ▶ That ring is a subring of K

Classifying ring substructures

If you can exhibit $\text{RING-PROP}_{R/K}$ in your graphical language over K -bits

▶ That ring is a subring of K

▶ The interpretation is determined by $\left[\begin{array}{c} | \\ \textcircled{0} \end{array} \right]$ and $\left[\begin{array}{c} | \\ \textcircled{1} \end{array} \right]$

Questions

Any questions on $\text{RING-PROP}_{R/K}$?

Full substructure

What about $\text{RING-PROP}_{K/K}$?

Universality

Is $\text{RING-PROP}_{K/K}$ universal?

Universality

Is $\text{RING-PROP}_{K/K}$ universal?

No. Can't form morphisms of shape $1 \rightarrow 0$

Compact closed

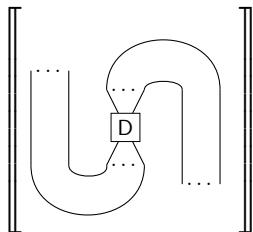
How about if we make $\text{RING-PROP}_{K/K}$ compact closed?

Compact closed

Define $\text{RING-PROP}_{R/K}^T$ as $\text{RING-PROP}_{R/K}$ with cups and caps.

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The diagram shows a box labeled D with a cup and cap. The box is enclosed in a double-line frame. The cup is on the left and the cap is on the right. The box is labeled D in the center. The diagram is followed by the equation $:= \llbracket D \rrbracket^T$.

Back over fields

This requirement on cups and caps in $\text{RING-PROP}_{R/K}^T$ forces the usual interpretations.

$$\llbracket \text{cup} \rrbracket = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \llbracket \text{cap} \rrbracket = (1 \ 0 \ 0 \ 1)$$

Universality

Is $\text{RING-PROP}_{K/K}^T$ universal?

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Yes. Can form all of RING_K .

Uniqueness

Every aspect of the interpretation of $\text{RING-PROP}_{R/K}^T$ is entirely determined up to choice of basis, and it's universal

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It has some sort of uniqueness property.

Phase-Rings as Monoids

The category **PR-GC**(R/K) of Phase-Ring Graphical Calculi (exhibited as monoids) for a ring R interpreted into K -bits has:

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$$(G, \llbracket \cdot \rrbracket, \{g_r\}_{r \in R}, g_+, g_\times)$$

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Need addition and multiplication to be sound, and the interpretation to be faithful.

Phase-Rings as Monoids

A morphism

$$f : (G, [\cdot], \{g_r\}_{r \in R}, g_+, g_\times) \rightarrow (H, [\cdot], \{h_r\}_{r \in R}, h_+, h_\times)$$

is a morphism of compact closed props

$$f : G\text{-diagrams} \rightarrow H\text{-diagrams}$$

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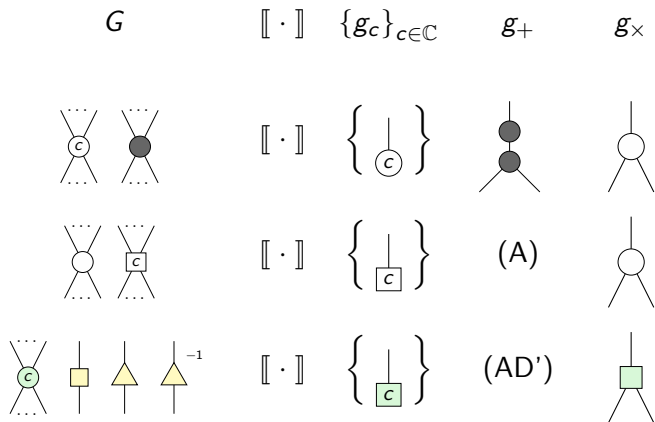
$$f(g_r) = h_r \qquad f(g_+) = h_+ \qquad f(g_\times) = h_\times$$

What's in this category?

Looking at **PR-GC**(\mathbb{C}/\mathbb{C}), it includes

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The morphism out of RING_K is easy to construct:

$$f\left(\begin{array}{c} | \\ \circ \\ k \end{array}\right) := h_k$$

$$f\left(\begin{array}{c} | \\ \square \\ + \\ \diagdown \quad \diagup \end{array}\right) := h_+$$

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The morphism is unique because these are all the generators

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Its interpretation is fully determined (up to change of basis)

Summary

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Summary

- ▶ RING_R is a universal, complete calculus for R -bits, just built from ring operations acting on states
- ▶ The substructures $\text{RING-PROP}_{R/K}$ have their interpretations determined up to change of basis, and R must be a subring of K
- ▶ $\text{RING}_{\mathbb{C}}$ is initial in **PR-GC**(\mathbb{C}/\mathbb{C})

Thank you!